Supplementary material for the PRL submitted paper of "Violation of Wiedemann-Franz law at the Kondo breakdown quantum critical point": Quantum Boltzman equation study

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In this supplementary material we evaluate electrical conductivity based on the quantum Boltzman equation near the Kondo breakdown quantum critical point. We find that the quantum Boltzman equation study gives an exactly same result with the diagrammatic study in the so called decoupling limit.

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I. ELIASHBERG FRAMEWORK

To study universal physics near the QCP, it is necessary to go beyond the mean-field description. Actually, fluctuation corrections turned out to be important for thermodynamics near the Kondo breakdown QCP, resulting from both hybridization and gauge fluctuations [1].

We start from

\[
\mathcal{L}_{ALM} = \mathcal{L}_e + \mathcal{L}_f + \mathcal{L}_b + \mathcal{L}_a + \mathcal{L}_K,
\]

\[
\mathcal{L}_e = e^*_\alpha (\partial_\tau - \mu_e) c_\sigma + \frac{1}{2m_c} |\partial_\tau c_\sigma|^2,
\]

\[
\mathcal{L}_f = f_\sigma^*(\partial_\tau - \mu_f - i\alpha \tau) f_\sigma + \frac{1}{2m_f} |(\partial_\tau - i\alpha) f_\sigma|^2,
\]

\[
\mathcal{L}_b = b^*(\partial_\tau - i\alpha) b + \frac{1}{2m_b} |(\partial_\tau - i\alpha) b|^2,
\]

\[
\mathcal{L}_a = \frac{1}{4g^2} f_{\mu\nu} f_{\mu\nu},
\]

\[
\mathcal{L}_K = V(b^* e^*_\alpha f_\sigma + H.c.),
\]

where this continuum Lagrangian can be derived from Eq. (1) in our submitted manuscript. \(\mu_e\) and \(m_c \approx (2t)^{-1}\) are chemical potential and band mass for conduction electrons, respectively. \(\mu_f\) and \(m_f = m_c/\alpha\) are those for spinons, where \(\alpha = J_\chi/t\) is an effective ratio between the bandwidth of conduction electrons and that of spinons, chosen to be \(\alpha \approx O(10^{-2} \sim 10^{-3}) < 1\). \(m_b = (2N^2\rho_c)^{-1}\) is band mass for holons, resulting from high energy fluctuations of conduction electrons and spinons. The Lagrange multiplier field \(\lambda\) is replaced with the time component of \(U(1)\) gauge field \(a_\tau\) for clarity. Spatial components of gauge fluctuations arise from the phase factor of the hopping parameter in the representation of \(\chi_{ij} = \chi e^{ia_{ij}}\) with nearest neighbor sites \(i,j\). The Maxwell term for bare dynamics of gauge fluctuations results from high energy fluctuations of both spinons and holons with an effective coupling strength \(g\), where \(f_{\mu\nu} = \partial_\nu a_\mu - \partial_\mu a_\nu\) is the field strength tensor.

Four kinds of field variables, that is, conduction electrons, spinons, holons and \(U(1)\) gauge fields make the effective Lagrangian complicated. It is not easy to treat such degrees of freedom self-consistently. Recently, it was explicitly demonstrated that Eliashberg framework is the minimal self-consistent treatment for an effective field theory near its quantum critical point [2]. The Eliashberg treatment neglects momentum dependence of fermion self-energies and vertex corrections. The first assumption is based on the fact that momentum dependence of fermion self-energies is regular, safely replaced with the Fermi momentum, and singular physics arises from their frequency dependence [3]. This can be checked explicitly at least in the one loop approximation. The second assumption is more serious than the first one, sometimes called Migdal theorem [4]. When fermions are much faster than bosons, vertex corrections can be neglected since the pre-factor in the renormalized vertex is given by the ratio of fermion and boson velocities. However, this turns out to be not sufficient for the Eliashberg framework. Another parameter is shown to be needed, that is, the fermion flavor number \(N\). In the large \(N\) limit the Eliashberg framework is justified [2].

In the Eliashberg framework both hybridization and gauge fluctuations are taken into account self-consistently.
where the effective action is found to be

\[
S_{\text{eff}} = T \sum_{\omega} \int \frac{d^3 k}{(2\pi)^3} \left[ \epsilon ( -i \omega - \mu_c + \epsilon_k + \Sigma_c (i\omega) ) g_{\sigma} + \Gamma_{\sigma} ( -i \omega - \mu_f + \epsilon_k + \Sigma_f (i\omega) ) f_{\sigma} \right] + T \sum_{\omega} \int \frac{d^3 q}{(2\pi)^3} \left[ b^\dagger ( -i \Omega + \epsilon_q + \Pi_b (q, i\Omega) ) b + \left( \frac{\Omega^2 + q^2}{2g^2} + \Pi_a (q, i\Omega) \right) a_{\mu} ( \delta_{\mu\nu} - \frac{q_{\nu} q_{\sigma}}{q^2} ) a_{\sigma} \right].
\]

Here, the self-energy correction for conduction electrons originates from scattering with hybridization fluctuations, given by

\[
\Sigma_c (i\omega) = -V^2 T \sum_{\omega} \int \frac{d^3 q}{(2\pi)^3} \frac{D_b (q, i\Omega) G_f (k_{\parallel}^f + q, i\omega + i\Omega),}{G(k_{\parallel}^f + q, i\omega + i\Omega)}
\]

where \( G_f (k, i\omega) = \left( -i \omega - \epsilon_k - \Sigma_f (i\omega) \right)^{-1} \) and \( D_b (q, i\Omega) = \left( -i \omega - \epsilon_q - \Pi_b (q, i\Omega) \right)^{-1} \) are fully renormalized propagators with bare bands \( \epsilon_k^f = \frac{k^2}{2m_f} \) and \( \epsilon_q = \frac{q^2}{2m} \) for spinons and holons, respectively. Notice that the fermion self-energy depends on only frequency and there are no vertex corrections.

The self-energy correction for spinons arises from scattering with both hybridization and gauge fluctuations, given by

\[
\Sigma_f (i\omega) = \Sigma_f^b (i\omega) + \Sigma_f^a (i\omega),
\]

\[
\Sigma_f^b (i\omega) = -V^2 T \sum_{\omega} \int \frac{d^3 q}{(2\pi)^3} \frac{D_b (q, i\Omega) G_c (k_{\parallel}^f - q, i\omega - i\Omega),}{G(k_{\parallel}^f - q, i\omega - i\Omega)}
\]

\[
\Sigma_f^a (i\omega) = -T \sum_{\omega} \sum_{\omega'} \int \frac{d^3 q}{(2\pi)^3} F(q, k) D_a (q, i\Omega) G_f (k_{\parallel}^f + q, i\omega + i\Omega),
\]

where \( F(q, k) \equiv \frac{1}{2} \sum_{i,j=1}^2 v_i^f \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) v_j^f \) is the bare current-vertex with the spinon velocity \( v_i^f = \frac{k_i + q_i/2}{m_j} \), and \( G_c (k, i\omega) = \left( -i \omega - \epsilon_k + \Sigma_c (i\omega) \right)^{-1} \) and \( D_a (q, i\Omega) = \left( \frac{q^2 + q^2}{2g^2} + \Pi_a (q, i\Omega) \right)^{-1} \) are fully renormalized propagators for conduction electrons and gauge fluctuations, respectively.

Self-energy corrections for holon and gauge fluctuations are given by

\[
\Pi_b (q, i\Omega) = NV^2 T \sum_{\omega} \int \frac{d^3 k}{(2\pi)^3} \frac{G_c (k, i\omega) G_f (k + q, i\omega + i\Omega)}{G(k + q, i\omega + i\Omega)} + T \sum_{i\Omega'} \int \frac{d^3 q'}{(2\pi)^3} \frac{B(q, q') D_a (q', i\Omega') D_b (q + q', i\Omega + i\Omega')}{B(q, q') D_a (q', i\Omega') D_b (q + q', i\Omega + i\Omega')},
\]

\[
\Pi_a (q, i\Omega) = -NT \sum_{i\Omega'} \int \frac{d^3 k}{(2\pi)^3} \frac{F(q, k) G_f (k, i\omega) G_f (k + q, i\omega + i\Omega)}{G(k + q, i\omega + i\Omega)} - T \sum_{i\Omega'} \int \frac{d^3 q'}{(2\pi)^3} \frac{B(q, q') D_a (q', i\Omega') D_b (q + q', i\Omega + i\Omega')}{B(q, q') D_a (q', i\Omega') D_b (q + q', i\Omega + i\Omega')},
\]

where the holon self-energy results from both electron-spinon polarization and scattering with gauge fluctuations, and the self-energy for gauge fluctuations arises from current-current correlations of both spinons and holons. Here, \( B(q, k) \equiv \frac{1}{2} \sum_{i,j=1}^2 v_i^b \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) v_j^b \) is the holon-gauge current-vertex with the holon velocity \( v_i^b = \frac{k_i + q_i/2}{m_j} \).

The effective action Eq. (2) with self-energy corrections of Eqs. (3), (4), and (5) defines the Eliashberg framework for the Kondo breakdown QCP in the Anderson lattice model. Later, we evaluate all of these expressions for transport coefficients.

II. DIAGRAMMATIC STUDY

We first consider a diagrammatic approach for the electrical conductivity. Later, we compare this diagrammatic result with the quantum Boltzmann equation study.

A. Contribution of conduction electrons

It is straightforward to evaluate all current-current correlation functions in the one loop approximation. For the electrical conductivity from conduction electrons, scattering with hybridization fluctuations relaxes electrical currents
of conduction electrons, imposed in the self-energy correction. An important point is that vertex corrections for such scattering processes can be neglected because the lowest order non-vanishing vertex correction is order of $\alpha^2/N$ [6], where $\alpha = J_N/t$ is the effective ratio for each fermion bandwidth, thus pretty much small even in the physical case of $N = 2$. Physically, the absence of vertex corrections for scattering with hybridization fluctuations results from heavy mass of spinons. In this respect transport time can be safely replaced with scattering time for such scattering mechanism. Actually, we will see the absence of vertex corrections for the electron conductivity in a certain limit of the quantum Boltzman equation approach.

The current-current correlation function for conduction electrons is given by
\[
\pi_c(i\Omega) = Nv_F^2 T \sum_{i\omega} \frac{d^3k}{(2\pi)^3} G_c(k, i\omega)G_c(k, i\omega + i\Omega),
\]
where $v_F$ is the Fermi velocity of conduction electrons, and $G_c(k, i\omega) = [i\omega - v_F^c k - \Sigma_c(k^c_F, i\omega)]^{-1}$ is the renormalized propagator of conduction electrons with the linearized dispersion near the Fermi surface.

Considering the dc conductivity $\sigma_c(T) = -\lim_{\Omega \to 0} \frac{\pi_c(i\Omega)}{3\pi^2(\Omega+i\delta)}$, we find
\[
\sigma_c(T) = C \rho_c v_F^c [3\Sigma_c(T)]^{-1},
\]
where $C = \frac{N}{\pi} \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{y^2 + 1}}$ is a positive numerical constant. We note that the quasiparticle life time $\tau_{qc}^{-1} = [3\Sigma_c(T)]^{-1}$ appears instead of the transport time for the electrical conductivity of conduction electrons.

For the self-energy correction of conduction electrons, we need a fully renormalized propagator for holon excitations. Using renormalized propagators for conduction electrons and spinons, we find the holon propagator
\[
D_h^{-1}(q, i\Omega) = -\left( \frac{q^2}{2m_h} + c \frac{|\Omega|}{q} \right),
\]
where $m_h = (2NV^2\rho_c)^{-1}$ is band mass of holons and $c = (2v_F^c m_h)^{-1}$ is a Landau damping coefficient. For this calculation, we have used the renormalized spinon propagator with a linearized dispersion near the Fermi surface, i.e., $G_f(k, i\omega) = [i\omega - v_F^f(k + q^*) - \Sigma_f(k^f_F, i\omega)]^{-1}$ like the electron propagator, where $q^* = |k^f_F - k^c_F|$ is the mismatch between each fermion Fermi-momenta, $k^f_F$ and $k^c_F$. We note that this expression remains unchanged in the Eliashberg framework even if we use bare propagators without self-energy corrections for fermion excitations. As discussed in the introduction, electron-spinon polarization is Landau-damped at $E > E^*$, but reduced to the propagator of the dilute Bose gas model at $E < E^*$. An important point is that most spectral weight is focused on the $z = 3$ regime, allowing us to safely ignore the $z = 2$ contribution [5, 6].

Using the renormalized holon and spinon propagators, we find the electron self-energy
\[
\Sigma_c(T > E^*) = \frac{m_h V^2}{12\pi v_F} T \ln \left( \frac{2T}{E^*} \right),
\]
\[
\Sigma_c(T < E^*) = \frac{m_h V^2}{12\pi v_F} \frac{T^2}{E^*} \ln 2,
\]
consistent with $z = 3$ scaling except the logarithmic correction at $T > E^*$ and recovering the Fermi liquid behavior at $T < E^*$. This self-energy correction is protected from vertex corrections because $z = 3$ criticality for hybridization fluctuations supports Migdal theorem, where on-shell (or resonance) fermion momenta are much larger than on-shell boson one at the same energy, implying that bosons are much slower than fermions [2]. As a result, we find the dc conductivity for conduction electrons
\[
\sigma_c(T > E^*) = C \rho_c v_F^c [\frac{m_h V^2}{12\pi v_F} T \ln \left( \frac{2T}{E^*} \right)]^{-1},
\]
\[
\sigma_c(T < E^*) = C \rho_c v_F^c \left[ \frac{m_h V^2}{12\pi v_F} \frac{T^2}{E^*} \ln 2 \right]^{-1}.
\]

B. Contribution of spinons

The spinon conductivity is found to be $\sigma_f(T) = C \rho_f v_F^c [3\Sigma_f(T)]^{-1}$ in the one loop approximation as the case for conduction electrons, where the spinon self-energy results from scattering with both hybridization and gauge
fluctuations, given by \( \Sigma_f(i\omega) = \Sigma_f^h(i\omega) + \Sigma_f^s(i\omega) \). The self-energy correction due to hybridization fluctuations is

\[
\Sigma^h_f(T > E^*) = \frac{m_f v_f^2}{12\pi v_F^2} T \ln \left( \frac{2T}{E^*} \right),
\]

\[
\Sigma^h_f(T < E^*) = \frac{m_f v_f^2}{12\pi v_F^2} \ln 2, \quad (11)
\]

basically the same form as that for conduction electrons except \( v_F^\prime \rightarrow v_F \).

The self-energy correction due to gauge fluctuations is the same form as \( \Sigma^h_f(T > E^*) \) because the gauge propagator is also governed by \( z = 3 \) criticality, resulting from Landau damping of spinons and holons. Actually, it is given by

\[
D_a^{-1}(q, i\Omega) = -\left( \frac{g^2}{2m_a} + \frac{\Omega}{q} \right) \quad (12)
\]
in the Eliashberg framework, where \( c_a \approx \pi N(2m_f v_F)^{-1} \) is the Landau damping coefficient, and \( m_a \approx m_f/N \) plays the role of band mass for gauge fluctuations. Then, we find \( \Sigma^h_f(T) = \frac{m_f v_f^2}{2\pi v_F^2} T \ln \left( \frac{2T}{E^*_G} \right) \), where \( E_G \) is the low energy cutoff for gauge fluctuations, playing the same role as \( E^* \) for hybridization fluctuations. Unfortunately, \( E_G \) vanishes at the Kondo breakdown QCP, resulting in divergence of the spinon self-energy at finite temperatures. Accordingly, the spinon conductivity vanishes in the quantum critical regime. However, this should be regarded as an artifact of gauge non-invariance for the spinon self-energy. Gauge invariance is crucial for the spinon conductivity.

The gauge invariance can be incorporated by vertex corrections [7–9], here for the spinon-current vertex. The current-current correlation function with vertex corrections is given by

\[
\pi_f(i\Omega) = T \sum_{k, \omega} \frac{d^3k}{(2\pi)^3} v_f' \cdot \Gamma(i\omega, i\omega + i\Omega) G_f(k, i\omega) G_f(k, i\omega + i\Omega)
\]

\[
\approx v_f'^2 T \sum_{k, \omega} \int \frac{d^3k}{(2\pi)^3} \gamma(i\omega, i\omega + i\Omega) G_f(k, i\omega) G_f(k, i\omega + i\Omega). \quad (13)
\]

\( \Gamma(i\omega, i\omega + i\Omega) \approx v_f'^2 \gamma(i\omega, i\omega + i\Omega) \) is a vector vertex for the current correlator, where \( \gamma(i\omega, i\omega + i\Omega) = 1 + \Lambda(i\omega, i\omega + i\Omega) \) is a scalar vertex with its correction part \( \Lambda(i\omega, i\omega + i\Omega) \). \( \left[ G_f(k, i\omega) \right]^{-1} = \left[ g_f^h(k, i\omega) \right]^{-1} - \Sigma_f^h(i\omega) \) is a full propagator for spinons, including self-energy corrections from both hybridization and gauge fluctuations, where \( \left[ g_f^h(k, i\omega) \right]^{-1} = \left[ g_f(k, i\omega) \right]^{-1} \) is a spinon propagator with only the self-energy correction due to hybridization fluctuations.

Expanding the full propagator as \( G_f(k, i\omega) \approx g_f^h(k, i\omega) + [g_f^h(k, i\omega)]^2 \Sigma_f^h(i\omega) + \ldots \), we find \( \pi_f(i\Omega) \approx \pi_f^h(i\Omega) + \pi_f^s(i\Omega) \), where the full vertex \( \Lambda(i\omega, i\omega + i\Omega) \) is replaced with the gauge-current vertex \( \Lambda_g(i\omega, i\omega + i\Omega) \), and each correlation function is given by

\[
\pi_f^h(i\Omega) \approx v_f'^2 T \sum_{k, \omega} \int \frac{d^3k}{(2\pi)^3} g_f^h(k, i\omega) g_f^h(k, i\omega + i\Omega),
\]

\[
\pi_f^s(i\Omega) \approx v_f'^2 T \sum_{k, \omega} \int \frac{d^3k}{(2\pi)^3} \left\{ [g_f^h(k, i\omega)]^2 \Sigma_f^h(i\omega) g_f^h(k, i\omega + i\Omega) + \ldots \right\}.
\]

As explicitly shown by these equations, the contribution from hybridization fluctuations has no vertex corrections, consistent with the conductivity of conduction electrons while the correlation function associated with gauge fluctuations has vertex corrections, indeed. As a result, divergence in the self-energy correction due to gauge fluctuations is cancelled by that in the vertex correction, giving rise to a gauge invariant finite result, multiplied by an additional \( 1 - \cos \theta \) factor in the expression for the self-energy correction, where \( \theta \) is an angle between the initial and final momenta [8]. In other words, the quasiparticle life time due to gauge fluctuations is replaced with the transport time.

Actually, we find \( \sigma_f^h(T) \approx C\rho_f v_F' \tau_f^h,sc(T) \) with the scattering time \( \tau_f^h,sc(T) \equiv [\Sigma_f^h(T)]^{-1} \) in the spinon conductivity.
of $\sigma_f(T) = \sigma_f^g(T) + \sigma_f^h(T)$ while the gauge-correction part turns out to be

$$\sigma_f^g(T) \approx C \rho_F v_F^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) \left\{ -\frac{3 \Sigma_f^g(\omega)}{[3 \Sigma_f^g(\omega)]^2} + \frac{3 \Sigma_f^h(\omega)}{[3 \Sigma_f^h(\omega)]^2} + \ldots \right\}$$

$$\approx C \rho_F v_F^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) \left\{ -\frac{3 \Sigma_f^g(\omega)}{[3 \Sigma_f^g(\omega)]^2} + \frac{3 \Sigma_f^h(\omega)}{[3 \Sigma_f^h(\omega)]^2} - \frac{3 \Sigma_{f,tr}(\omega)}{[3 \Sigma_f^h(\omega)]^2} + \ldots \right\}$$

$$\approx -C \rho_F v_F^2 \tau_{f,sc}^2(T) \tau_{f,tr}^2(T).$$

Here, $\tau_{f,tr}^2(T) \equiv [3 \Sigma_{f,tr}(T)]^{-1}$ is the transport time, where $\Sigma_{f,tr}(\omega) = 3 \Sigma_f^g(\omega) - 3 \Sigma_f^h(\omega)$ is the self-energy correction with the $1 - \cos \theta$ factor, and $\Sigma_f^h(\omega)$ has the $\cos \theta$ factor in the self-energy expression. The transport time is found to be

$$[\tau_{f,sc}(T)]^{-1} = \left( \frac{k_F^2}{16\pi N} \right) \gamma_a T^{\frac{2}{3}},$$

where $\gamma_a = 2 m_a e_a / V^2$ is the Landau damping coefficient for gauge fluctuations. The $T^{5/3}$ behavior is well known in the gauge theory context [10].

One cautious person may propose that the above perturbation result appears from the following non-perturbation expression for the gauge invariant spinon conductivity

$$\sigma_f(T) = \frac{C \rho_F v_F^2}{\tau_{f,sc}(T)^{-1} + [\tau_{f,tr}(T)]^{-1}},$$

basically the same as $\sigma_f(T) = \frac{C \rho_F v_F^2}{3 \Sigma_f^g(T) + 3 \Sigma_f^h(T)}$, but the scattering time for gauge fluctuations $[\tau_{f,sc}(T)]^{-1} \equiv 3 \Sigma_f^g(T)$ is replaced with the transport time $[\tau_{f,tr}(T)]^{-1}$ as a result of the gauge invariance. Actually, this non-perturbation expression can be justified, performing an infinite-order summation for vertex corrections.

Considering the following expression for the gauge invariant correlation function of spinon currents

$$\pi_f(i\Omega) \approx v_F^2 T \sum_{\omega} \frac{d^3 k}{(2\pi)^3} \left\{ G_f(k, i\omega) G_f(k, i\omega + i\Omega) + \Lambda_q(i\omega, i\omega + i\Omega) G_f(k, i\omega) G_f(k, i\omega + i\Omega) \right\},$$

we find

$$\sigma_f(T) \approx \frac{C \rho_F v_F^2}{\tau_{f,sc}(T)^{-1} + [\tau_{f,tr}(T)]^{-1}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) \left\{ \frac{1}{3 \Sigma_f^g(\omega)} + \frac{3 \Sigma_f^h(\omega)}{[3 \Sigma_f^h(\omega)]^2} + \ldots \right\}$$

$$= \frac{C \rho_F v_F^2}{\tau_{f,sc}(T)^{-1} + [\tau_{f,tr}(T)]^{-1}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) \left\{ \frac{1}{3 \Sigma_f^g(\omega)} + \frac{\Sigma_f^h(\omega)}{[3 \Sigma_f^h(\omega)]^2} + \ldots \right\}.$$

Recalling $3 \Sigma_f^g(\omega) - 3 \Sigma_f^h(\omega) = 3 \Sigma_f^g(\omega) + 3 \Sigma_{f,tr}(\omega)$, we find the non-perturbation expression for the gauge invariant spinon conductivity. We can justify this non-perturbation result based on the quantum Boltzmann equation approach in the decoupling limit.

We emphasize that $[\tau_{f,sc}(T)]^{-1} \propto T \ln(2T/E^*)$ is much larger than $[\tau_{f,tr}(T)]^{-1} \propto T^{5/3}$, thus gauge-fluctuation corrections can be neglected safely. This supports that $z = 3$ hybridization fluctuation is the main scattering source for transport phenomena near the Kondo breakdown QCP although both gauge and hybridization fluctuations are important for thermodynamics [1].

In summary, the spinon conductivity is given by

$$\sigma_f(T > E^*) = \frac{C \rho_F v_F^2}{m_F V^2 T \ln\left( \frac{2T}{E^*} \right) + \left( \frac{m_F v_F^2}{16\pi N} \right) \gamma_a T^{\frac{5}{3}}}$$

$$\sigma_f(T < E^*) = \frac{C \rho_F v_F^2}{m_F V^2 T^2 \ln 2 + \left( \frac{m_F v_F^2}{16\pi N} \right) \gamma_a T^{\frac{5}{3}}}.$$

(16)
C. Contribution of holons

The correlation function of electrical currents for holon excitations is given by

\[
\pi_b(i\Omega) = T \sum_{i\Omega'} \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{2D_b(q, i\Omega') D_b(q, i\Omega) D_b(q, i\Omega + i\Omega')},
\]

where \( v_b = \left( \frac{1}{m_b} - \frac{e^{Q' + \Omega/2}}{q} \right) \) is the holon velocity for \( z = 3 \) dynamics. The renormalized holon propagator \( D_b^{-1}(q, i\Omega) = -\left( \frac{q^2}{2m_b} + \frac{e^{i\Omega}}{q} \right) \) is utilized for the correlation function, where \( z = 3 \) dynamics due to Landau damping of conduction electrons and spinons plays an important role while gauge-fluctuation corrections can be ignored owing to their sub-dominance.

As a result, we find the transport coefficient for holon excitations

\[
\sigma_b(T > E^*) = C_b(2m_b c)^\frac{1}{2} \left( \frac{T}{E^*} \right)^\frac{1}{4} \left( \frac{T}{E^*} \right)^\frac{3}{4},
\]

\[
\sigma_b(T < E^*) = C_b(2m_b c)^\frac{1}{2} \left( 1 - \frac{1}{24} \right) \left( \frac{T^2}{E^*} \right)^\frac{1}{2},
\]

where \( C_b = \frac{1}{\pi^2} \int_1^\infty dy \left( 1 - \frac{1}{\pi^2} \right)^2 \frac{1}{y^2} \) is a positive numerical constant.

III. QUANTUM BOLTZMAN EQUATION STUDY

We reexamine electrical transport near the Kondo breakdown QCP based on quantum Boltzman equation. Since gauge invariance is automatically incorporated in quantum Boltzman equation [11], we are expected to understand irrelevance or relevance of vertex corrections for transport coefficients, associated with both hybridization and gauge fluctuations. In this study we assume that both hybridization and gauge fluctuations are in equilibrium, and consider only fermion contributions, consistent with the one-loop result for the transport coefficient. Since we have two kinds of fermion excitations, we find coupled quantum Boltzman equations for distributions of conduction electrons and spinons. Solving such coupled quantum Boltzman equations, we find that the diagrammatic result is recovered in the decoupling limit of these equations, where vertex corrections for scattering with hybridization fluctuations can be ignored, but those for scattering with gauge fluctuations should be introduced in the spinon conductivity.

A. Application to U(1) gauge theory for a spin liquid state

We apply the quantum Boltzman equation to the transport problem of U(1) gauge theory, showing that our treatment successfully recovers the known result,

\[
S = \int_0^\beta d\tau \int d^d r \left[ \psi^\dagger_\sigma \left( \partial_\tau - i a_\tau - \mu_\psi \right) \psi_\sigma + \frac{1}{2m_\psi} \psi^\dagger_\sigma \left( -i \partial_i - a_i \right)^2 \psi_\sigma \right] + \int \frac{dv}{2\pi} \sum_q D(q, \nu) \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) a_i(q, \nu) a_j(-q, -\nu),
\]

where \( D(q, \nu) = \left( -i \gamma_\psi \frac{q^2}{q^4} + \chi_\psi q^2 \right)^{-1} \) is the gauge propagator with the diamagnetic susceptibility \( \chi_\psi \) and Landau damping coefficient \( \gamma_\psi \).

Compared with the effective field theory for the Kondo breakdown QCP of the Anderson lattice model, the above U(1) gauge theory is a simplified version since it does not have both holons and conduction electrons. In this section we focus on the mathematical structure, in particular, the gauge invariant expression for conductivity [12, 13] instead of the physical aspect, in order to prepare for the Boltzman equation study of the Anderson lattice model.

We rewrite the quantum Boltzman equation as

\[
[\partial_\omega f(\omega)] \Theta(k, \omega) [A(k, \omega)]^2 \mathbf{v}_k \cdot \mathbf{E} = I_{\text{coll}}(k, \omega),
\]
where the collision term is given by
\[ I_{\text{coll}}(k, \omega) = 2i\Gamma(k, \omega)G^\text{<}(k, \omega) - i\Sigma^\text{<}(k, \omega)A(k, \omega). \] (21)

Here, we used the identity of
\[
\Sigma^\text{>}(k, \omega)G^\text{<}(k, \omega) - \Sigma^\text{<}(k, \omega)G^\text{>}(k, \omega) = 2i\Gamma(k, \omega)G^\text{<}(k, \omega) - i\Sigma^\text{<}(k, \omega)A(k, \omega),
\]
where the lesser self-energy is given by
\[
\Sigma^\text{<}(k, \omega) = \sum_q \int_0^\infty \frac{dv}{\pi} \left| \frac{k \times \hat{q}}{m_\psi} \right|^2 \Im D(q, \nu)[(n(\nu) + 1)G^\text{<}(k + q, \omega + \nu) + n(\nu)G^\text{<}(k + q, \omega - \nu)].
\] (22)

Since we are considering linear response, we expand the lesser Green function up to the first order for an electric field
\[
G^\text{<}(k, \omega) = iA(k, \omega) \left[ f(\omega) - \frac{\partial f(\omega)}{\partial \omega} \right] E \cdot v_k \Lambda(k, \omega). \] (23)

Inserting this ansatz into the lesser self-energy, we obtain the following expression for the lesser self-energy
\[
\Sigma^\text{<}(k, \omega) = i \sum_q \int_0^\infty \frac{dv}{\pi} \left| \frac{k \times \hat{q}}{m_\psi} \right|^2 \Im D(q, \nu) \left\{ (n(\nu) + 1)\left[ f(\omega + \nu)A(k + q, \omega + \nu) + n(\nu)f(\omega - \nu)A(k + q, \omega - \nu) \right] \right\}
\]
\[+ i \sum_q \int_0^\infty \frac{dv}{\pi} \left| \frac{k \times \hat{q}}{m_\psi} \right|^2 \Im D(q, \nu)E \cdot v_{k+q} \left\{ (n(\nu) + 1)\left( -\frac{\partial f(\omega + \nu)}{\partial \omega} \right)A(k + q, \omega + \nu)\Lambda(k + q, \omega + \nu) \right\}.
\] (24)

We introduce the following identities for thermal factors of fermions and bosons,
\[
\{n(\nu) + 1\}f(\omega + \nu) = f(\omega)[n(\nu) + f(\omega + \nu)],
\]
\[
n(\nu)f(\omega - \nu) = -f(\omega)[n(\nu) + f(\omega - \nu)],
\] (25)

and
\[
\{n(\nu) + 1\}\left( -\frac{\partial f(\omega + \nu)}{\partial \omega} \right) = \{n(\nu) + f(\omega + \nu)\} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} \left( -\frac{\partial f(\omega)}{\partial \omega} \right),
\]
\[
n(\nu)\left( -\frac{\partial f(\omega - \nu)}{\partial \omega} \right) = -\{n(\nu) + f(\omega - \nu)\} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} \left( -\frac{\partial f(\omega)}{\partial \omega} \right).
\] (26)

Then, the lesser self-energy becomes
\[
\Sigma^\text{<}(k, \omega) = i \sum_q \int_0^\infty \frac{dv}{\pi} \left| \frac{k \times \hat{q}}{m_\psi} \right|^2 \Im D(q, \nu)f(\omega) \left\{ \{n(\nu) + f(\omega + \nu)\}A(k + q, \omega + \nu) - \{n(\nu) + f(\omega - \nu)\}A(k + q, \omega - \nu) \right\}
\]
\[+ i \sum_q \int_0^\infty \frac{dv}{\pi} \left| \frac{k \times \hat{q}}{m_\psi} \right|^2 \Im D(q, \nu)E \cdot v_{k+q} \left\{ (n(\nu) + 1)\left( -\frac{\partial f(\omega + \nu)}{\partial \omega} \right) \right\} \{n(\nu) + f(\omega + \nu)\} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A(k + q, \omega + \nu)\Lambda(k + q, \omega + \nu)
\]
\[= \{n(\nu) + f(\omega - \nu)\} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} A(k + q, \omega - \nu)\Lambda(k + q, \omega - \nu) \}.
\] (27)

Inserting both the lesser Green function and self-energy into the quantum Boltzmann equation, we find
\[
\{ A(k, \omega) \}^2 \partial_\omega f(\omega)E \cdot v_k \Gamma(k, \omega) = 2i\Gamma(k, \omega)iA(k, \omega) \left\{ \left( \frac{\partial f(\omega)}{\partial \omega} \right)E \cdot v_k \Lambda(k, \omega) \right\}
\]
\[-iA(k, \omega) \left[ i \sum_q \int_0^\infty \frac{dv}{\pi} \left| \frac{k \times \hat{q}}{m_\psi} \right|^2 \Im D(q, \nu)f(\omega) \left\{ \{n(\nu) + f(\omega + \nu)\}A(k + q, \omega + \nu) - \{n(\nu) + f(\omega - \nu)\}A(k + q, \omega - \nu) \right\}
\]
\[+ i \sum_q \int_0^\infty \frac{dv}{\pi} \left| \frac{k \times \hat{q}}{m_\psi} \right|^2 \Im D(q, \nu)E \cdot v_{k+q} \left\{ (n(\nu) + 1)\left( -\frac{\partial f(\omega + \nu)}{\partial \omega} \right) \right\} \{n(\nu) + f(\omega + \nu)\} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A(k + q, \omega + \nu)\Lambda(k + q, \omega + \nu)
\]
\[-\{n(\nu) + f(\omega - \nu)\} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} A(k + q, \omega - \nu)\Lambda(k + q, \omega - \nu) \}.
\] (28)
Noting that the imaginary part of the self-energy or scattering rate is given by
\[ 2 \Gamma(k, \omega) = \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| \frac{k \times \hat{q}}{m_0} \right|^2 \Im D(q, \nu) \left\{ \{n(\nu) + f(\omega + \nu)\} A(k + q, \omega + \nu) - \{n(-\nu) + f(\omega - \nu)\} A(k + q, \omega - \nu) \right\}, \]
\[ (29) \]
the above expression is simplified as
\[ \Lambda(k, \omega) = \frac{1}{2} A(k, \omega) + \frac{1}{2 \Gamma(k, \omega)} \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| \frac{k \times \hat{q}}{m_0} \right|^2 \Im D(q, \nu) \]
\[ \left\{ \{n(\nu) + f(\omega + \nu)\} A(k + q, \omega + \nu) - \{n(-\nu) + f(\omega - \nu)\} A(k + q, \omega - \nu) \right\} \left( \frac{v_k \cdot v_{k+q}}{v_k^2} \right), \]
\[ (30) \]
where we divide the resulting equation with \([-\partial_x f(\omega)] \mathbf{E} \cdot \mathbf{v}_k\).

Since usual transport phenomena occur near the Fermi surface except some topological quantities such as Hall conductivity, we consider the following approximation
\[ \Lambda(k_F, \omega) \approx \frac{1}{2} A(k_F, \omega) + \frac{1}{2 \Gamma(k_F, \omega)} \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| \frac{k \times \hat{q}}{m_0} \right|^2 \Im D(q, \nu) \]
\[ \left\{ \{n(\nu) + f(\omega + \nu)\} A(k_F + q, \omega + \nu) - \{n(-\nu) + f(\omega - \nu)\} A(k_F + q, \omega - \nu) \right\} \left( \frac{v_{k_F} \cdot v_{k_F+q}}{v_{k_F}^2} \right) \Lambda(k_F, \omega), \]
\[ (31) \]
where the momentum is replaced with the Fermi momentum \(k_F\) and frequency dependence in both the "vertex-distribution" function \(\Lambda(k_F, \omega)\) and thermal Fermi factor is simplified. This approximation will be justified by the fact that it gives rise to the known result in the gauge theory context.

Introducing the relative angle \(\theta\) between the initial \(k_F\) and final \(k_F + q\) momenta, we see
\[ \Lambda(k_F, \omega) = \frac{1}{2} A(k_F, \omega) + \frac{1}{2 \Gamma(k_F, \omega)} \frac{N_F}{2\pi} \int d\xi \int_{-1}^1 d\cos \theta \int_0^\infty \frac{d\nu}{\pi} |v_{k_F}^2 \cos^2(\theta/2)| \Im D(q, \nu) \cos \theta \]
\[ \left\{ \{n(\nu) + f(\omega + \nu)\} A(k_F + q, \omega + \nu) - \{n(-\nu) + f(\omega - \nu)\} A(k_F + q, \omega - \nu) \right\} \Lambda(k_F, \omega). \]
\[ (32) \]
Then, the vertex-distribution function can be written as follows
\[ \Lambda(k_F, \omega) = \frac{2 \Gamma(k_F, \omega)}{2 \Gamma_1 - \cos(k_F, \omega)} A(k_F, \omega), \]
\[ (33) \]
where
\[ 2 \Gamma_1 - \cos(k_F, \omega) = \frac{N_F}{2\pi} \int d\xi \int_{-1}^1 d\cos \theta \int_0^\infty \frac{d\nu}{\pi} |v_{k_F}^2 \cos^2(\theta/2)| \Im D(q, \nu) [1 - \cos \theta] \]
\[ \left\{ \{n(\nu) + f(\omega + \nu)\} A(k_F + q, \omega + \nu) - \{n(-\nu) + f(\omega - \nu)\} A(k_F + q, \omega - \nu) \right\}. \]
\[ (34) \]
Note that there appears \(1 - \cos \theta\) factor in \(\Gamma_1 - \cos(k_F, \omega)\). In this respect \([2 \Gamma_1 - \cos(k_F, \omega)]^{-1}\) is identified with the transport time \(\tau_T(\omega)\), capturing large angle scattering dominantly.

The electrical (charge) or number conductivity is expressed by the lesser Green function,
\[ J_{\mu} = -i \int \frac{d^3k}{(2\pi)^3} \frac{k_\mu k}{m} \int \frac{d\omega}{2\pi} G^{<}(k, \omega) \]
\[ = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} A(k, \omega) v_k^\mu \left[ f(\omega) - \left( \frac{\partial f(\omega)}{\partial \omega} \right) v_k^\mu E_\nu \Lambda(k, \omega) \right], \]
\[ (35) \]
where the near-equilibrium ansatz for the lesser Green function is inserted in the last equality. The first term corresponds to the contribution of equilibrium, which does not generate currents. The second contribution gives rise to the following expression for the conductivity
\[ \sigma_{\mu\nu}(T) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} v_k^\mu v_k^\nu \left( \frac{\partial f(\omega)}{\partial \omega} \right) A(k, \omega) \Lambda(k, \omega). \]
\[ (36) \]
Here, $\Lambda(k, \omega)$ includes vertex corrections in the Kubo formulæ.

Inserting the vertex-distribution function into the conductivity expression, we find

$$\sigma(T) = v_F^2 \frac{N_F}{2\pi} \int d\xi \int_{-1}^{1} d\cos \theta \int \frac{d\omega}{2\pi} \left( -\frac{\partial f(\omega)}{\partial\omega} \right) \frac{2\Gamma(k_F, \omega)}{2\Gamma_{1-\cos(k_F, \omega)}} |A_F(\xi, \omega)|^2,$$

(37)

where the momentum integration is replaced with the energy and angle ones near the Fermi surface. Performing the integration, we reach the final expression of the conductivity

$$\sigma(T) \approx C N_F v_F^2 \tau_{tr}(T)$$

(38)

with $C = \frac{\Delta}{v_F} \int_{-\infty}^{\infty} dy \frac{1}{(y^2 + 1)^{7/2}}$, where the transport time is $\tau_{tr}(T) = [2\Gamma_{1-\cos}(T)]^{-1}$, as emphasized before.

The transport time turns out to be $\tau_{tr}(T) \propto T^{-5/3}$, giving rise to $\sigma(T) \propto T^{-5/3}$, completely consistent with the previous study [12, 13]. An important point is that although the self-energy correction due to gauge fluctuations is diverging at finite temperatures, the gauge invariant expression for the conductivity allows only the finite result, cancelling the divergence via the vertex correction. $1-\cos \theta$ reflects this point as discussed in the diagrammatic study. This is the power of the quantum Boltzman equation, imposing the vertex correction naturally.

The present formulation differs from the previous approach in the fact that we did not decompose the gauge field as the study of Refs. [12, 13], where the low energy gauge field giving rise to divergence is neglected and only high energy gauge fluctuations are taken. Although the vertex-distribution function itself is not well defined because its part corresponding to the scattering rate is divergent at finite temperatures, we showed that such decomposition is not necessary because the formal divergence should be cancelled in the last gauge invariant physical expression. This spirit goes exactly through that of the diagrammatic study.

### B. Application to the Kondo breakdown QCP of the Anderson lattice model

In the Kondo breakdown scenario we have four kinds of field variables, corresponding to conduction electrons, spinons, holons (hybridization fluctuations), and gauge bosons (collective spin fluctuations). Our main assumption for the transport study based on the quantum Boltzman equation approach is that both hybridization and gauge fluctuations are in equilibrium. This assumption is justified by the diagrammatic study, where contributions from boson excitations are small, compared with fermion contributions. In this respect we are allowed to have two coupled quantum Boltzman equations,

$$[A_c(k, \omega)]^2 \partial_\omega f(\omega) \mathbf{E} \cdot \mathbf{v}_c^F \Gamma_c(k, \omega) = I_{coll}^c(k, \omega),$$

$$I_{coll}^c(k, \omega) = 2i\Gamma_c(k, \omega) G_{c}^F(k, \omega) - i\Sigma_c(k, \omega) A_c(k, \omega)$$

(39)

for conduction electrons and

$$[A_f(k, \omega)]^2 \partial_\omega f(\omega) \mathbf{E} \cdot \mathbf{v}_f^F \Gamma_f(k, \omega) = I_{coll}^f(k, \omega),$$

$$I_{coll}^f(k, \omega) = 2i\Gamma_f(k, \omega) G_{f}^F(k, \omega) - i\Sigma_f(k, \omega) A_f(k, \omega)$$

(40)

for spinons.

Solving these coupled quantum Boltzman equations, we show that exactly the same result as the diagrammatic study is recovered in the decoupling limit of the Boltzman equations, where gauge invariance is automatically incorporated. However, vertex corrections arise beyond the decoupling limit, modifying the transport coefficient. We will discuss physics of vertex corrections in detail.

#### 1. Contribution of conduction electrons

The lesser self-energy for conduction electrons arises from scattering with hybridization fluctuations, given by

$$\Sigma_c^F(k, \omega) = V^2 \sum_q \int_0^{\infty} \frac{dv}{\pi} \Im D_h(q, v)[\{n(v) + 1\} G_{c}^F(k + q, \omega + v) + n(v) G_{c}^F(k + q, \omega - v)].$$

(41)

Since the spinon Green function appears in the electron self-energy, the two quantum Boltzman equations are coupled with each other. This coupling effect is the main character for the quantum Boltzman equation of the Anderson lattice model.
Inserting the following expression for the spinon lesser Green function

\[ G_f^-(k, \omega) = i A_f(k, \omega) \left[ f(\omega) - \left( \frac{\partial f(\omega)}{\partial \omega} \right) \mathbf{E} \cdot \mathbf{v}_f^k \Lambda_f(k, \omega) \right] \]  

(42)

into the electron lesser self-energy, we obtain

\[ \Sigma^\varepsilon_c(k, \omega) = i V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) f(\omega) \left\{ \{ n(\nu) + f(\omega + \nu) \} A_f(k + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k + q, \omega - \nu) \right\} \]

\[ + i V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \mathbf{E} \cdot \mathbf{v}_f^k \left( - \frac{\partial f(\omega)}{\partial \omega} \right) \left\{ \{ n(\nu) + f(\omega + \nu) \} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_f(k + q, \omega + \nu) \right. 

\[ \left. - \{ n(-\nu) + f(\omega - \nu) \} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} A_f(k + q, \omega - \nu) \right\}, \]

(43)

where the identities for fermion and boson thermal factors are utilized. The quantum Boltzmann equation for conduction electrons becomes

\[ [A_c(k, \omega)]^2 \frac{\partial f(\omega)}{\partial \omega} \mathbf{E} \cdot \mathbf{v}_c^k \Gamma_c(k, \omega) \]

\[ = -2\Gamma_c(k, \omega) A_c(k, \omega) \left\{ f(\omega) - \left( \frac{\partial f(\omega)}{\partial \omega} \right) \mathbf{E} \cdot \mathbf{v}_c^k \Lambda_c(k, \omega) \right\} \]

\[ + A_c(k, \omega) \left\{ V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) f(\omega) \left\{ \{ n(\nu) + f(\omega + \nu) \} A_f(k + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k + q, \omega - \nu) \right\} \right. 

\[ \left. + V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \mathbf{E} \cdot \mathbf{v}_k^{k+q} \left( - \frac{\partial f(\omega)}{\partial \omega} \right) \left\{ \{ n(\nu) + f(\omega + \nu) \} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_f(k + q, \omega + \nu) \right. 

\[ \left. - \{ n(-\nu) + f(\omega - \nu) \} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} A_f(k + q, \omega - \nu) \right\} \right\}. \]

(44)

Considering the scattering rate

\[ 2\Gamma_c(k, \omega) = V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \left\{ \{ n(\nu) + f(\omega + \nu) \} A_f(k + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k + q, \omega - \nu) \right\}, \]

(45)

the above equation is simplified as

\[ \Lambda_c(k, \omega) = \frac{1}{2} A_c(k, \omega) + \frac{V^2}{2\Gamma_c(k, \omega)} \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \left\{ \{ n(\nu) + f(\omega + \nu) \} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_f(k + q, \omega + \nu) \right. 

\[ - \{ n(-\nu) + f(\omega - \nu) \} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} A_f(k + q, \omega - \nu) \right\} \left( \frac{\mathbf{v}_k^{k+q} \cdot \mathbf{v}_c^k}{\mathbf{v}_c^k \cdot \mathbf{v}_c^k} \right), \]

(46)

where the resulting equation is divided by the $[-\partial_c f(\omega)]\mathbf{E} \cdot \mathbf{v}_c^k$ factor.

Taking transport near the Fermi momentum, this expression can be approximated as

\[ \Lambda_c(k_F^c, \omega) \approx \frac{1}{2} A_c(k_F^c, \omega) + \frac{V^2}{2\Gamma_c(k_F^c, \omega)} \frac{N_F^c}{2\pi} \int d\xi \int_{-1}^1 d \cos \theta_{cf} \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \left( \frac{v_f^k}{v_F^c} \cos \theta_{cf} \right) \left\{ \{ n(\nu) + f(\omega + \nu) \} A_f(k_F^c + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k_F^c + q, \omega - \nu) \right\} \Lambda_f(k_F^c, \omega), \]

(47)

where an angle variable $\theta_{cf}$ between $k_F^c + q$ and $k_F^c$ is introduced. This approximation is exactly the same as that of the U(1) gauge theory, discussed in the previous section. It is important to notice that the vertex-distribution function for conduction electrons is related with that for spinons. We should know the vertex-distribution function for spinons.
2. Contribution of spinons

The lesser self-energy for spinon excitations results from scattering with both hybridization and gauge fluctuations, given by

\[ \Sigma^\gamma_f(k, \omega) = \Sigma^{b<}_f(k, \omega) + \Sigma^{a<}_f(k, \omega), \]

\[ \Sigma^{b<}_f(k, \omega) = V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_0(q, \nu) \{ n(\nu) + 1 \} G^\gamma_e(k + q, \omega + \nu) + n(\nu) G^\gamma_e(k + q, \omega - \nu), \]

\[ \Sigma^{a<}_f(k, \omega) = \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| k \times \hat{q} \right|^2 \Im D_0(q, \nu) \{ n(\nu) + 1 \} G^\gamma_f(k + q, \omega + \nu) + n(\nu) G^\gamma_f(k + q, \omega - \nu). \]  

(48)

Inserting the lesser Green function for conduction electrons

\[ G^\gamma_e(k, \omega) = i A_e(k, \omega) \left[ f(\omega) - \left( \frac{\partial f(\omega)}{\partial \omega} \right) \right] \]

with that for spinons into the above expression for the lesser self-energy, we find the self-energy correction due to hybridization fluctuations

\[ \Sigma^{b<}_f(k, \omega) = i V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_0(q, \nu) f(\omega) \{ n(\nu) + f(\omega + \nu) \} A_e(k + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_e(k + q, \omega - \nu) \]

\[ + i V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_0(q, \nu) E \cdot \mathbf{\nu}_{k+q} \{ n(\nu) + f(\omega + \nu) \} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_e(k + q, \omega + \nu) \]

\[ - \{ n(-\nu) + f(\omega - \nu) \} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} A_e(k + q, \omega - \nu) \]  

(50)

and that due to gauge fluctuations

\[ \Sigma^{a<}_f(k, \omega) = i \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| k \times \hat{q} \right|^2 \Im D_0(q, \nu) f(\omega) \{ n(\nu) + f(\omega + \nu) \} A_f(k + q, \omega + \nu) \]

\[ - \{ n(-\nu) + f(\omega - \nu) \} A_f(k + q, \omega - \nu) \]  

\[ + i \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| k \times \hat{q} \right|^2 \Im D_0(q, \nu) E \cdot \mathbf{\nu}_{k+q} \{ n(\nu) + f(\omega + \nu) \} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_f(k + q, \omega + \nu) \]

\[ - \{ n(-\nu) + f(\omega - \nu) \} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} A_f(k + q, \omega - \nu) \]  

(51)

where the identities for fermion and boson thermal factors are utilized.

Inserting these expressions with the near-equilibrium Green function of spinons into the quantum Boltzmann equation
of the spinon sector, we find the following expression

\[
[A_f(k, \omega)]^2 \partial_\omega f(\omega) \mathbf{E} \cdot \mathbf{v}_k^f \Gamma_f(k, \omega) = -2\Gamma_f(k, \omega) A_f(k, \omega) \left\{ f(\omega) - \left( \frac{\partial f(\omega)}{\partial \omega} \right) \mathbf{E} \cdot \mathbf{v}_k^f A_f(k, \omega) \right\} + A_f(k, \omega) \left[ V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) f(\omega) \right] \\
\left\{ n(\nu) + f(\omega + \nu) A_c(k + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_c(k + q, \omega - \nu) \right\} + V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \mathbf{E} \cdot \mathbf{v}_k^{f+q} \left( -\frac{\partial f(\omega)}{\partial \omega} \right) \left\{ n(\nu) + f(\omega + \nu) \right\} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_c(k + q, \omega + \nu) A_c(k + q, \omega - \nu) \\
- \{ n(-\nu) + f(\omega - \nu) \} \frac{1 - f(\omega - \nu)}{1 - f(\omega)} A_f(k + q, \omega - \nu) \right\} + \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| k \times \hat{\nu} \right|^2 \Im D_a(q, \nu) f(\omega) \left\{ n(\nu) + f(\omega + \nu) \right\} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_f(k + q, \omega + \nu) A_f(k + q, \omega - \nu) \right\}. \tag{52}
\]

Considering that the scattering rate for spinons is given by

\[
2\Gamma_f(k, \omega) = V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \left\{ n(\nu) + f(\omega + \nu) A_c(k + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_c(k + q, \omega - \nu) \right\} + \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| k \times \hat{\nu} \right|^2 \Im D_a(q, \nu) \mathbf{v}_k^{f+q} \left\{ n(\nu) + f(\omega + \nu) \right\} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_f(k + q, \omega + \nu) A_f(k + q, \omega - \nu) \right\}, \tag{53}
\]

the quantum Boltzmann equation becomes

\[
\mathbf{v}_k^f A_f(k, \omega) = \frac{v_k^f}{2} A_f(k, \omega) + \left[ V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \mathbf{v}_k^{f+q} \left\{ n(\nu) + f(\omega + \nu) \right\} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_c(k + q, \omega + \nu) A_c(k + q, \omega - \nu) \right\} \\
+ \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| k \times \hat{\nu} \right|^2 \Im D_a(q, \nu) \mathbf{v}_k^{f+q} \left\{ n(\nu) + f(\omega + \nu) \right\} \frac{1 - f(\omega + \nu)}{1 - f(\omega)} A_f(k + q, \omega + \nu) A_f(k + q, \omega - \nu) \right\}. \tag{54}
\]

Taking transport near the Fermi surface, this expression can be rewritten as follows

\[
\mathbf{v}_k^f A_f(k^f_F, \omega) \approx \frac{v_k^f}{2} A_f(k^f_F, \omega) + \frac{1}{2\Gamma_f(k^f_F, \omega)} \left[ V^2 \sum_q \int_0^\infty \frac{d\nu}{\pi} \Im D_b(q, \nu) \mathbf{v}_k^{f+q} \left\{ n(\nu) + f(\omega + \nu) \right\} A_c(k^f_F + q, \omega + \nu) A_c(k^f_F + q, \omega - \nu) \right\} \\
- \{ n(-\nu) + f(\omega - \nu) \} A_c(k^f_F + q, \omega + \nu) + \sum_q \int_0^\infty \frac{d\nu}{\pi} \left| k^f_F \times \hat{\nu} \right|^2 \Im D_a(q, \nu) \mathbf{v}_k^{f+q} \left\{ n(\nu) + f(\omega + \nu) \right\} A_f(k^f_F + q, \omega + \nu) A_f(k^f_F + q, \omega - \nu) \right\}, \tag{55}
\]

basically the same approximation as that of the gauge theory in the previous section.
Introducing angle variables, we see

\[ \Lambda_f(k_F^f, \omega) = \frac{1}{2} A_f(k_F^f, \omega) + \frac{1}{2 \pi} \left[ V^2 \frac{N_F^c}{2 \pi} \int \frac{d \xi}{-1} d \cos \theta_{cf} \left( \frac{v_F^c}{v_F} \cos \theta_{cf} \right) \right] \]

\[ \int_0^\infty \frac{d \nu}{\pi} \Im D_b(q, \nu) \left\{ \{ n(\nu) + f(\omega + \nu) \} A_c(k_F^c + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_c(k_F^c + q, \omega - \nu) \right\} \Lambda_c(k_F^c, \omega) \]

\[ + \frac{N_F^f}{2 \pi} \int \frac{d \xi}{-1} d \cos \theta_{ff} \cos \theta_{ff} \int_0^\infty \frac{d \nu}{\pi} \left[ \nu f^2 \cos^2(\theta_{ff}/2) \right] \Im D_a(q, \nu) \left\{ \{ n(\nu) + f(\omega + \nu) \} A_f(k_F^f + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k_F^f + q, \omega - \nu) \right\} \Lambda_f(k_F^f, \omega) \].

(56)

In the quantum Boltzmann equation for conduction electrons, we found the relation between the vertex-distribution functions of conduction electrons and spinons. Inserting this into the above, we find

\[ \Lambda_f(k_F^f, \omega) = \frac{1}{2} A_f(k_F^f, \omega) + \frac{1}{2 \pi} \left[ V^2 \frac{N_F^c}{2 \pi} \int \frac{d \xi}{-1} d \cos \theta_{cf} \left( \frac{v_F^c}{v_F} \cos \theta_{cf} \right) \right] \]

\[ \int_0^\infty \frac{d \nu}{\pi} \Im D_b(q, \nu) \left\{ \{ n(\nu) + f(\omega + \nu) \} A_c(k_F^c + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_c(k_F^c + q, \omega - \nu) \right\} \Lambda_c(k_F^c, \omega) \]

\[ + \frac{N_F^f}{2 \pi} \int \frac{d \xi}{-1} d \cos \theta_{ff} \cos \theta_{ff} \int_0^\infty \frac{d \nu}{\pi} \left[ \nu f^2 \cos^2(\theta_{ff}/2) \right] \Im D_a(q, \nu) \left\{ \{ n(\nu) + f(\omega + \nu) \} A_f(k_F^f + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k_F^f + q, \omega - \nu) \right\} \Lambda_f(k_F^f, \omega) \].

(57)

Introducing the following scattering functions

\[ 2 \Gamma_{f, \cos}(k_F^f, \omega) \equiv V^2 \frac{N_F^c}{2 \pi} \int \frac{d \xi}{-1} d \cos \theta_{cf} \left( \frac{v_F^c}{v_F} \cos \theta_{cf} \right) \int_0^\infty \frac{d \nu}{\pi} \Im D_b(q, \nu) \]

\[ \left\{ \{ n(\nu) + f(\omega + \nu) \} A_c(k_F^c + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_c(k_F^c + q, \omega - \nu) \right\}, \]

\[ 2 \Gamma_{g, \cos}(k_F^f, \omega) \equiv \frac{N_F^f}{2 \pi} \int \frac{d \xi}{-1} d \cos \theta_{ff} \cos \theta_{ff} \int_0^\infty \frac{d \nu}{\pi} \left[ \nu f^2 \cos^2(\theta_{ff}/2) \right] \Im D_a(q, \nu) \]

\[ \left\{ \{ n(\nu) + f(\omega + \nu) \} A_f(k_F^f + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k_F^f + q, \omega - \nu) \right\}, \]

\[ 2 \Gamma_{c, \cos}(k_F^f, \omega) \equiv V^2 \frac{N_F^c}{2 \pi} \int \frac{d \xi}{-1} d \cos \theta_{cf} \left( \frac{v_F^c}{v_F} \cos \theta_{cf} \right) \int_0^\infty \frac{d \nu}{\pi} \Im D_b(q, \nu) \]

\[ \left\{ \{ n(\nu) + f(\omega + \nu) \} A_c(k_F^c + q, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k_F^f + q, \omega - \nu) \right\}, \]

(58)

the vertex-distribution function for spinons can be simplified as follows

\[ \Lambda_f(k_F^f, \omega) = \frac{1}{2} \left\{ A_f(k_F^f, \omega) + \frac{\Gamma_{f, \cos}(k_F^f, \omega)}{\Gamma_f(k_F^f, \omega)} A_c(k_F^c, \omega) \right\} + \left\{ \frac{\Gamma_{b, \cos}(k_F^f, \omega)}{\Gamma_f(k_F^f, \omega)} \right\} \frac{\Gamma_{c, \cos}(k_F^c, \omega)}{\Gamma_f(k_F^f, \omega)} \Lambda_f(k_F^f, \omega). \]

(59)

As a result, we find

\[ \Lambda_f(k_F^f, \omega) = \frac{1}{2} \left\{ 1 + \Gamma_{f, \cos}(k_F^f, \omega) \right\} \frac{\Gamma_{c, \cos}(k_F^c, \omega)}{\Gamma_f(k_F^f, \omega)} \Lambda_f(k_F^f, \omega) \]

(60)
where

\[ 2\Gamma_f^c(k_F, \omega) \equiv V^2 N_F^c \int d\xi \int_{-1}^{1} d\cos \theta \epsilon_f \int_{0}^{\infty} \frac{d\nu}{\pi} \Im D_\nu(q, \nu) \]

\[ \{ \{ n(\nu) + f(\omega + \nu) \} A_c(k_F^c, \nu, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_c(k_F^c, \nu, \omega - \nu) \}, \]

\[ 2\Gamma_f^c,1-\cos(k_F, \omega) \equiv \frac{N_F^c}{2\pi} \int d\xi \int_{-1}^{1} d\cos \theta \epsilon_f \int_{0}^{\infty} \frac{d\nu}{\pi} [v_f^2 \cos^2(\theta f/2)] \Im D_\nu(q, \nu) \]

\[ \{ \{ n(\nu) + f(\omega + \nu) \} A_f(k_F^f, \nu, \omega + \nu) - \{ n(-\nu) + f(\omega - \nu) \} A_f(k_F^f, \nu, \omega - \nu) \}. \]  

3. Conductivity in the decoupling limit

In the vertex-distribution function for spinons, Eq. (55), we neglect the coupling term \( \Lambda_c(k_F^c, \omega) \) as the zeroth order approximation for the transport study. Then, we find

\[ \Lambda_f(k_F^f, \omega) = \frac{1}{2} A_f(k_F^f, \omega) + \frac{\Gamma_f^{c, \cos}(k_F^c, \omega)}{\Gamma_f(k_F^f, \omega)} \Lambda_f(k_F^f, \omega), \]

giving rise to

\[ \Lambda_f(k_F^f, \omega) = \frac{1}{2} \frac{\Gamma_f(k_F^f, \omega) A_f(k_F^f, \omega)}{\Gamma_f(k_F^f, \omega) + \Gamma_{f,1-\cos}(k_F^f, \omega)}. \]

Inserting this expression into the spinon conductivity, we find

\[ \sigma_f(T) = v_F^2 \int_{-1}^{1} d\cos \theta \int_{0}^{\infty} \frac{d\nu}{\pi} \Im D_\nu(q, \nu) \]

\[ = \frac{C N_F^c v_F^2}{2\Gamma_f^c(T) + 2\Gamma_{f,1-\cos}(T)}. \]

This is exactly the same as that of the diagrammatic study.

The vertex-distribution function for conduction electrons becomes

\[ \Lambda_c(k_F^c, \omega) \approx \frac{1}{2} A_c(k_F^c, \omega) + \frac{1}{2} \frac{\Gamma_c^{c, \cos}(k_F^c, \omega)}{\Gamma_c(k_F^c, \omega)} \frac{\Gamma_f(k_F^f, \omega) A_f(k_F^f, \omega)}{\Gamma_f(k_F^f, \omega) + \Gamma_{f,1-\cos}(k_F^f, \omega)}. \]

Calling

\[ \frac{\Gamma_c^{c, \cos}(k_F^c, \omega)}{\Gamma_c(k_F^c, \omega)} = O(v_f/\nu_F) \approx \alpha \ll 1, \]

the second contribution in the electron vertex-distribution can be neglected. As a result, the conductivity from conduction electrons is free from its vertex correction, becoming

\[ \sigma_c(T) = v_F^2 \int_{-1}^{1} d\cos \theta \int_{0}^{\infty} \frac{d\nu}{\pi} \Im D_\nu(q, \nu) \approx \frac{C v_F^2}{2\Gamma_c^c(T)}, \]

which coincides with that of the diagrammatic study.